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A bayesian inference of material parameters from DIC data

M. Bhattacharyya, P. Feissel

Université de technologie de Compiègne, Laboratoire Roberval (FRE2012), Compiègne, France

Résumé — The framework of the study is to obtain deterministic and probabilistic approximations of the material parameters from measurement informations acquired from digital image correlation data. The deterministic process involves formulation of an optimal control approach. The probabilistic framework inculcates this optimal control method in a Bayesian inference framework. A Markov Chain Monte Carlo sampling method is applied to obtain the posterior probability density function, along with a radial basis function type interpolation for the numerical frugality of the samplings.
Mots clés — inverse problem, identification, Bayesian inference.

1 Introduction

Model updating is a fundamental issue in model-based applications such as numerical simulation, health monitoring, source identification, etc. Model updating classically consists in adjusting the model parameters in order to decrease some distance between the model prediction and the measurements. This is classically approached either deterministically or stochastically.

As far as the deterministic approach is concerned, key point is to develop or adapt identification strategies based on DIC measurements. Some specific methods have been proposed in the past years and a review can be found in [1]. These inverse approaches can be grouped into two large families [2] :

— Approaches by auxiliary fields : based on the weak form of equilibrium, specific choices for the test field lead to a direct identification of the sought parameters. Among these, we can cite the Virtual Fields Method (VFM) and the Reciprocity Gap Method (RGM) for full-field data.

— Approaches by minimization : from the overdetermined set of equation traducing the available information, some equations will be verified exactly through imposed constraints, while the other equations will simply be verified at best through the minimization of a gap to these equations. The following methods have been applied for full-field displacement data : the Finite Element Model Updating method (FEMU), the Equilibrium Gap Method (EGM), the Constitutive Relation Error method (CRE) and the Modified Constitutive Relation Error method (M-CRE).

Each inverse approach has its own advantage to use in specific application. However, there are two common problems : 1. Not all information is available. For example, in many cases, the full-field measurement is only on a sub-part of the specimen or boundary conditions may be unknown or not completely known. It leads to a lack of information outside the measurement zone or on the boundary of the specimen. Therefore, EGM which needs measurements over the whole domain, and FEMU/RGM which needs full boundary conditions to be performed, may have leave apart some of the available information or add supplementary hypothesis (e.g. on boundary conditions). 2. Not all the available information is reliable. For example, there are measurement uncertainties in experimental study and model errors in numerical analysis, which may lead to a loss of accuracy.

There is hence a need for identification methods allowing both the identification over the whole specimen, and the dealing of missing boundary condition. The Modified Constitutive Relation Error (M-CRE) addresses some of the aforementioned issues however it does not enforce the constitutive relation with reliability [3]. Therefore the optimal control method proposed here considers only the error with respect to the displacement field of the experiments and the model. This method consists of segregating the relations into reliable and less reliable sets, and it does not require complete information of the boundaries and measurement zone does not need to be on the complete structure.

To address the error originating from model deficiency or measurement noise stochastic analyses are necessary. In the present work, parametric model updating is investigated from a Bayesian perspective

by considering all model parameters to be updated as random variables [4]. Bayesian inference approach inculcates prior parametric information on the possible range of values using probability density functions. This approach provides a robust framework for both model and measurement errors. The objective of the Bayesian inference is to obtain the posterior probability density with the informations of the prior density and error in model and/or measurement. In this respect Monte Carlo Markov Chain (MCMC) methods is used based on Metropolis–Hastings algorithm to explore the posterior pdf [6].

One limitation of model updating with MCMC sampling is its high computational cost, especially when applied to large/complex finite element (FE) models since each sample requires a full computation of the latter. Reduction in computational cost is achieved by using Radial Basis Function (RBF) network [7] which circumvents the FE simulation by machine learning process.

2 Optimal control method

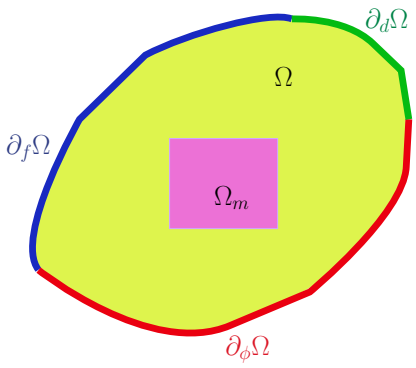

FIGURE 1 – Model of the identification domain

Consider a two-dimensional continuous elastic medium as shown in Figure 1 defined within domain Ω. The displacement field is measured within domain Ω_m, and the load is measured over $\partial_f\Omega$. A part of the boundary $\partial_d\Omega$ is considered to be free of load, and the information on rest of the boundary $\partial_\phi\Omega$ is considered to unknown. Now the governing equations can be segregated into two parts :

— Reliable equations :

$$\text{Equilibrium equation} \quad \underline{div}\left(\underline{\underline{\sigma}}\right) = \underline{0} \text{ in } \Omega \tag{1}$$

$$\text{Constitutive relation} \quad \underline{\underline{\sigma}} = \mathbf{C}(\underline{\theta})\,\underline{\underline{\varepsilon}} \text{ in } \Omega \tag{2}$$

$$\text{Kinematic compatibility condition} \quad \underline{\underline{\varepsilon}} = \frac{1}{2}\left(\underline{\underline{\nabla}}\,\underline{u} + \underline{\underline{\nabla}}^T\,\underline{u}\right) \text{ in } \Omega \tag{3}$$

$$\text{Free edge boundary} \quad \underline{\underline{\sigma}} \cdot \underline{n} = \underline{0} \text{ on } \partial_d\Omega \tag{4}$$

$$\text{Global load boundary} \quad \int_{\partial_f\Omega} \underline{\underline{\sigma}} \cdot \underline{n} dS \cdot \underline{n_0} = F_0 \text{ and } \int_{\partial_f\Omega} \underline{r} \times \underline{\underline{\sigma}} \cdot \underline{n} dS = \underline{M_0} \text{ on } \partial_f\Omega \tag{5}$$

where $\underline{\underline{\sigma}}$ is the Cauchy stress tensor, $\mathbf{C}$ is the Hooke's tensor depending on the material parameter $\underline{\theta}$. $\underline{\underline{\varepsilon}}$ is the infinitesimal strain tensor with $\underline{u}$ being the displacement field. $\underline{n}$ and $\underline{n_0}$ are unit normals on the surface, F_0 is the global force. $\underline{r}$ is the radial vector from a point of reference where the global moment $\underline{M_0}$ is measured.

— Less reliable equation :

$$\text{Displacement measurements} \quad \underline{u} = \underline{\tilde{u}} \text{ in } \Omega_m \tag{6}$$

where $\underline{\tilde{u}}$ is the measured displacement.

The formulation of the identification problem consists then in the confrontation of all the theoretical and experimental data in our disposal. The reliable equations are exactly verified and allow to define the kinematically, statically, constitutive relation admissible spaces, respectively $\mathcal{U}_{Ad}$, $\mathcal{S}_{Ad}$ and $\mathcal{C}_{Ad}$, such

that :

$$\mathcal{U}_{Ad} = \left\{ \underline{u} \in H^1(\Omega) \right\} \tag{7}$$

$$\mathcal{S}_{Ad} = \left\{ \underline{\underline{\sigma}} \in H^{div}(\Omega) / \underline{div}\left(\underline{\underline{\sigma}}\right) = \underline{0} \text{ in } \Omega,\ \underline{\underline{\sigma}} \cdot \underline{n} = \underline{0} \text{ on } \partial_d\Omega, \right.$$
$$\left. \int_{\partial_f\Omega} \underline{\underline{\sigma}} \cdot \underline{n} dS \cdot \underline{n_0} = F_0 \text{ and } \int_{\partial_f\Omega} \underline{r} \times \underline{\underline{\sigma}} \cdot \underline{n} dS = \underline{M_0} \text{ on } \partial_f\Omega \right\} \tag{8}$$

$$\mathcal{C}_{Ad} = \left\{ (\underline{u}, \underline{\underline{\sigma}}) \in H^1(\Omega) \times H^{div}(\Omega) / \underline{\underline{\sigma}} = \mathbf{C}(\underline{\theta})\, \underline{\underline{\varepsilon}} \right\} \tag{9}$$

Furthermore, an admissibility space θ_{Ad} is defined such that :

$$\theta_{Ad} = \{ \underline{\theta} \in \mathbb{R}^n / \mathbf{C}(\underline{\theta}) \text{ being symmetric positive definite} \} \tag{10}$$

The less reliable quantities are verified through minimisation of a functional $J(\underline{u})$

$$J(\underline{u}) = \frac{1}{2} \int_{\Omega_m} \|\underline{u} - \underline{\tilde{u}}\|^2 d\Omega_m \tag{11}$$

The identification problem then becomes

$$\text{Find } (\underline{u}, \underline{\theta}) \text{ that minimise } J(\underline{u}) \text{ under the constraint } \left(\underline{u}, \underline{\underline{\sigma}}, \underline{\theta}\right) \in ((\mathcal{U}_{Ad} \times \mathcal{S}_{Ad})) \cap \mathcal{C}_{Ad}) \times \theta_{Ad}$$

This problem is basically solved through a sequential minimisation. The problem is solved through two steps :

Basic problem It consists of solving for the displacement field for a fixed set of material parameters that conforms with the all the reliable governing equations and the measurement data.

$$\text{Find } \underline{u}, \text{minimising } J(\underline{u}), \text{under the contraints } \left(\underline{u}, \underline{\underline{\sigma}}\right) \in ((\mathcal{U}_{Ad} \times \mathcal{S}_{Ad})) \cap \mathcal{C}_{Ad})$$

Identification problem It consists of finding the optimal model parameter $\underline{\theta}^{opt}$ by minimising a cost function

$$G(\underline{\theta}) = J(\underline{u}(\underline{\theta})) \tag{12}$$

obtained from the resolution of the basic problem. This step basically translates into

$$\text{Find } \underline{\theta}^{opt}, \text{minimising } G, \text{under the contraint } \underline{\theta} \in \theta_{Ad}$$

2.1 Numerical implementation

For numerical implementation of the inverse problem, the discretisation of the continuous problem is achieved through finite element method. The discrete displacement vector U can be represented using shape function N as

$$\underline{u} = NU \tag{13}$$

The discretised form of the functional J is given as

$$J = \frac{1}{2} \left(\Pi U - \tilde{U}\right)^T \left(\Pi U - \tilde{U}\right) \tag{14}$$

where $\tilde{U}$ is the vector of measured displacement and Π is the extractor/projector operator which not only extracts U (defined within the Ω to fit the measurement zone Ω_m) but also projects the extracted nodal unknowns (from the finite element mesh to the measurement grid).

The global load boundary condition can be written as

$$U_0^{*T} K U = F_0,\ U_1^{*T} K U = M_0 \tag{15}$$

where U_0^* represents the normal vector, U_1^* represents the radial vector, K represents the stiffness matrix and F_0, M_0 indicates the global force and moment respectively. The equilibrium equation along with the constitutive relation within the interior nodes can be written as

$$K_{io} U = 0 \tag{16}$$

and the constitutive relation along with the free edge boundary can be written as

$$K_{do}U = 0 \tag{17}$$

The indices i, d and o represent degrees of freedom corresponding to interior, free edge and all nodes. The interior and free edge can be combined to a separate index g and the aforementioned equations can be combined into

$$K_{go}U = 0 \tag{18}$$

These constraints are enforced using Lagrange multipliers (Λ_l, Λ_m), which thereby leads to the functional

$$\mathcal{L} = \frac{1}{2}\left(\Pi U - \tilde{U}\right)^T\left(\Pi U - \tilde{U}\right) + \Lambda_l^T K_{go} + \Lambda_m^T\left(U_0^{*T} K U - F_0\right) + \Lambda_n^T\left(U_1^{*T} K U - F_0\right) \tag{19}$$

The stationarity of the problem associated to $\mathcal{L}$ leads to

$$\underbrace{\begin{bmatrix} \Pi^T\Pi & K_{og} & KU_0^* & KU_1^* \\ K_{go} & 0 & 0 & 0 \\ U_0^{*T}K & 0 & 0 & 0 \\ U_1^{*T}K & 0 & 0 & 0 \end{bmatrix}}_{M} \underbrace{\begin{pmatrix} U \\ \Lambda_l \\ \Lambda_m \\ \Lambda_n \end{pmatrix}}_{Y} = \underbrace{\begin{pmatrix} \Pi^T\tilde{U} \\ 0 \\ F_0 \\ M_0 \end{pmatrix}}_{S} \tag{20}$$

It is possible that M can be a rank-deficient matrix when $\Omega_m < \Omega$. In such a case eq (20) is solved using QR decomposition. This will provide the resolution of the basic problem.

For the identification of the material parameters, the cost function

$$G(\underline{\theta}) = J\left(U(\underline{\theta}), \underline{\theta}\right) \tag{21}$$

is minimised through simplex algorithm.

2.2 Numerical example

Let us consider a uniform homogeneous square plate as shown in Figure 2.

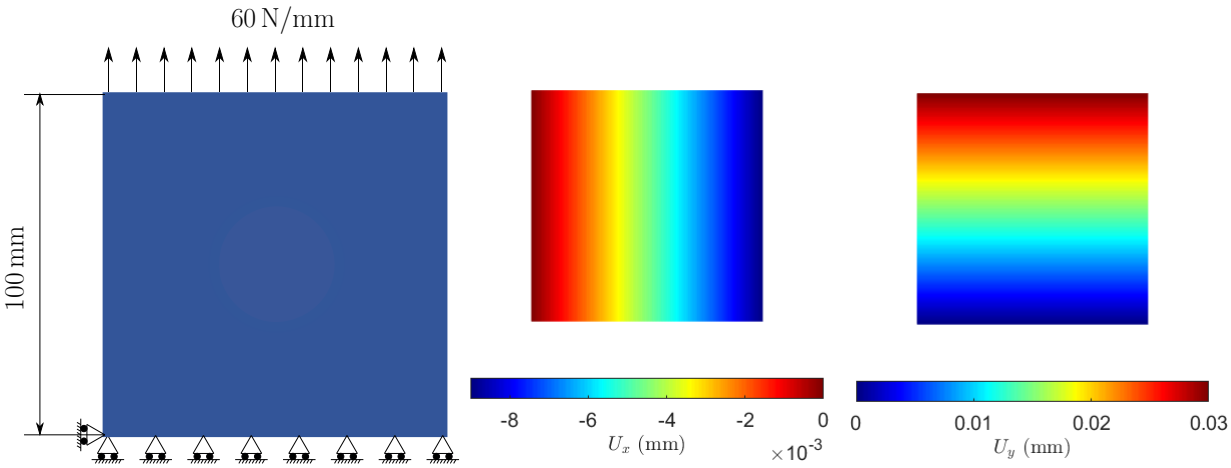

FIGURE 2 – Reference structure and the corresponding displacement fields

This reference problem is solved using classical FE using 40000 linear isoparametric quadrilateral elements consisting of a total 40401 nodes. The displacement field (Figure 2) obtained is considered to be a representative DIC measurement grid.

For the inverse problem, the global force will be 6000 N and global moment will be 0 N-mm. The top edge is considered to be global load boundary, the two side edges are considered to be free edges, and the bottom edge considered to have unknown boundary condition. The discretisation of the inverse problem is performed using 400 linear isoparametric quadrilateral elements consisting of a total 441 nodes. The material parameters $\underline{\theta}$ to be identified are the Lame's parameters λ and μ.

Several sets of inverse analyses are performed by changing the size of measurement zones. Each measurement zone is considered to be a square centred on the plate. For each size of the measurement zone Gaussian random white noises (percentage of the mean value of displacement) are added to the exact displacement fields in order to emulate measurement perturbation.

The error with respect to the reference material parameters defined by $\eta = \dfrac{\|\theta_{ref} - \theta\|}{\|\theta\|}$ is calculated in each of the cases and plotted in Figure 3. This error increases with decrease in the measurement zone and increases with increase in percentage of white noise.

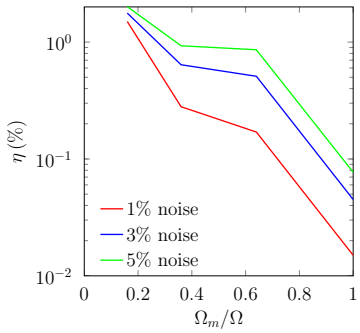

FIGURE 3 – Accuracy with respect to reference

3 Bayesian inference

The objective is basically to obtain posterior probability density function. Consider the Bayes theorem

$$p\left(\underline{\theta}/\tilde{U}\right) = \frac{p\left(\tilde{U}/\underline{\theta}\right) p\left(\underline{\theta}\right)}{p\left(\tilde{U}\right)} \tag{22}$$

Now if both the model and measurement error can be clubbed together i.e.

$$\delta U = \tilde{U} - U\left(\underline{\theta}\right) \tag{23}$$

with this assumption,

$$p\left(\tilde{U}/\underline{\theta}\right) = p_N\left(\tilde{U} - U\left(\underline{\theta}\right)\right) \tag{24}$$

The denominator is given as

$$p\left(\tilde{U}\right) = \int_{\Omega_\theta} p\left(\tilde{U}/\underline{\theta}\right) p\left(\underline{\theta}\right) d\theta \tag{25}$$

which can be represented as a constant parameter k. It is interesting to note that $p\left(\tilde{U}/\underline{\theta}\right) = p_N\left(\tilde{U} - U\left(\underline{\theta}\right)\right)$ is a function of the displacement field and the parameters (i.e. mean, variance and such others) describing the uncertainty is also in the kinematic field. However, the resolution of the basic problem provides a probability density function which is a function of the material parameters. This function is called the likelihood function $\mathcal{L}\left(\underline{\theta}\right)$ where

$$\mathcal{L}\left(\underline{\theta}\right) = p\left(\tilde{U}/\underline{\theta}\right) = p_N\left(\tilde{U} - U\left(\underline{\theta}\right)\right) \tag{26}$$

Thereby, (22) can be represented as

$$p\left(\underline{\theta}/\tilde{U}\right) = \frac{1}{k}\mathcal{L}\left(\underline{\theta}\right) p_\theta\left(\underline{\theta}\right) \propto \mathcal{L}\left(\underline{\theta}\right) p_\theta\left(\underline{\theta}\right) \tag{27}$$

where the likelihood function $\mathcal{L}$ is obtained from the solution of the basic problem with input uncertainty parameters (i.e. mean, variance and such others) prescribed within the kinematic space. The prior probability density p_θ describes the prior knowledge about the uncertainty of the material parameters through the uncertainty parameters (i.e. mean, variance and such others). The product of the likelihood and the prior probability is the target probability function. and to obtain the posterior probability $p\left(\underline{\theta}/\tilde{U}\right)$ is obtained through sampling technique known as Markov Chain Monte Carlo (MCMC) method.

In this context random walk Metropolis-Hastings algorithm is used to sample through the parametric space. The usefulness of the Metropolis-Hastings algorithm is the fact that exact equality of the target function is not required and just proportionality of the objective and target functions is enough to generate samples in the required probability function. This algorithm is given in Algorithm 1.

Algorithm 1 Metropolis-Hastings algorithm

1: Initialise $\underline{\theta}_0$
2: At iteration i
3: Find the the trial state $\underline{\theta}_i^* = \underline{\theta}_{i-1} + \delta\underline{\theta}$
4: Perturbation $\delta\underline{\theta} \backsim q(\delta\underline{\theta})$, where the proposal density $q(\delta\underline{\theta})$ is symmetric about zero
5: Calculate tolerance $\varepsilon \backsim \mathfrak{U}(0,1)$, where $\mathfrak{U}$ is uniform density
6: Calculate acceptance ratio $\alpha = \min\left(1, \dfrac{\mathcal{L}(\underline{\theta}_i^*)\, p_\theta(\underline{\theta}_i^*)}{\mathcal{L}(\underline{\theta}_{i-1})\, p_\theta(\underline{\theta}_{i-1})}\right)$
7: **if** $\varepsilon \leq \alpha$ **then**
8: $\underline{\theta}_i = \underline{\theta}_i^*$
9: **else**
10: $\underline{\theta}_i = \underline{\theta}_{i-1}$
11: **end if**
12: Set i=i+1, go to 2

It is clear from the Metropolis-Hastings algorithm that at each step of the MCMC process, there is a requirement of the resolution of the basic problem, which can be extremely expensive. For numerical frugality in this case, the basic problem is replaced with a Radial Basis Function (RBF) network, which is basically a two layer network that uses radial basis functions to approximate the functional value based on the informations of the training data set. So, the likelihood function is calculated at certain pre-chosen values in the parametric space, and thereafter using these values an RBF network is built that can approximate the likelihood at any values of the material parameters. The details of the RBF network can be found in .

After the samples are generated, the posterior probability density function can be estimated using kernel density estimation (KDE) which is basically a non-parametric density estimator. A kernel is a mathematical function that returns a probability for a given value of a random variable. The kernel effectively interpolates the probabilities across the range of outcomes for a random variable such that the sum of probabilities equals one. The kernel function weights the contribution of observations from a data sample based on their relationship or distance to a given query sample for which the probability is requested. A parameter, called the smoothing parameter or the bandwidth, controls the scope, or window of observations, from the data sample that contributes to estimating the probability for a given sample.

3.1 Numerical example

Considering the same structure as depicted in the deterministic problem, with the measurement zone completely encompassing the plate. A learning phase of 25×25 data points in the parametric space is used for building RBF network. A total of 2×10^5 MCMC samples have been used. The prior probability density function and the likelihood function are considered to be Gaussian in nature. Figure 4 and figure 5 represents the contour and surface plots for the posterior probability and the target function respectively, where an error of 6% is obtained. It also has to be mentioned that the proposal density of the Metropolis-Hastings algorithm and the bandwidth of the Kernel Density Estimation has to be tuned though trial and error to obtain an optimal final resolution.

Figure 6 shows the evolution of different quantities with respect to MCMC samples, the mean values of the parameters saturate to $\left\{6.62 \times 10^4, 7.69 \times 10^4\right\}$ MPa.

4 Conclusion

This research is an attempt to provide an optimal control approach for parametric identification with incomplete boundary information and with lack of knowledge of the complete measurement data. It also

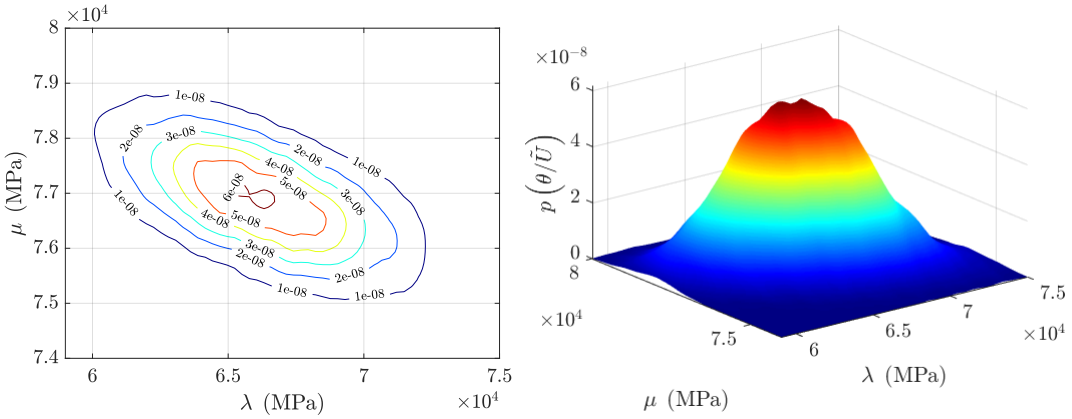

FIGURE 4 – Contour and surface plots for $p\left(\theta/\tilde{U}\right)$

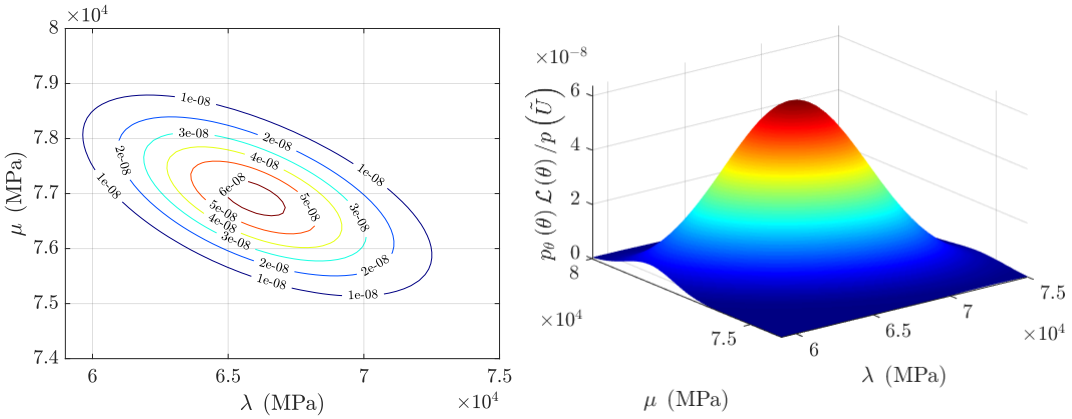

FIGURE 5 – Contour and surface plots for $\mathcal{L}\left(\theta\right) p_\theta\left(\theta\right)/p\left(\tilde{U}\right)$

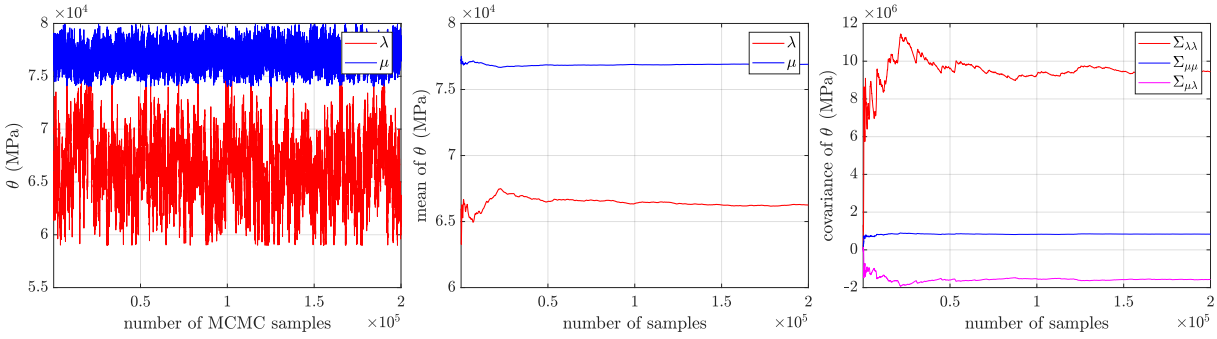

FIGURE 6 – Variation of quantities with respect to number of samples

provides a sampling method that gives a probabilistic description of the model parameters for stochastic errors that might arise due to model discrepancy or measurement perturbation.

In the future, the idea is to extend the probabilistic framework to address large number of model parameters. Also the future perspective is to extend the framework for non-linear material behaviour.

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