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Nitsche-XFEM for pressure jump interface condition in vibro-acoustic problems in presence of porous materials

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Résumé — We present a variational formulation based on the idea of Nitsche's method to account for thin layers in acoustic problems. Pressure jump model of robin-type condition is used to model the effects of thin acoustic layers. The computational domain is discretized using eXtended Finite Element Method (X-FEM) to represent the discontinuity in the elements and to mitigate the meshing restriction for complex geometry of interfaces. The porous materials with complex-valued propertis are also considered in current work, which introduce an energy dissipation in the system.
Mots clés — Nitsche's method, X-FEM, vibro-acoustic, porous materials.

1 Introduction

This work present a numerical method to simulate the sound propagation through absorption packages. Such sound system commonly consists of multiple porous layers coated by a relatively thin resistive film. Porous layers present an energy loss due to the interaction between poro-fluid and elastic frame. Equivalent fluid model as the JCA [1, 2] and Limp model [3] supposing the frame is motionless, and the dissipation in materials is considered by complex-valued properties. Biot's theory [4, 5] is another well-known model that introduce two coupled dynamic equations to describe the motion of solid and fluid phases. Thin film layers can be considered as a non-thickness interface with an appropriate interface conditions such as asymptotic model for elastic layers in scattering problems [6]. In this work, the most common pressure jump model [7] is adopted as the interface condition to describe sound behaviour of thin films.

The considered interface condition presents a strong discontinuity that is usually enforced with Lagrange multiplier within X-FEM [8]. This method leads to larger linear system to solve and the space of multiplier need to be chosen carefully to satisfy the "inf-sup" condition. Weak enforcement of such conditions can be also realized by Nitsche's method, which was first applied to solid mechanics [9]. And more recently, it has been extended in acoustic problems such as the work [10] where perfect interface condition is treated and the work [11] where an impedance interface in a homogeneous real-valued medium was investigated. Our work based on these former studies, we try to apply Nitsche's method within X-FEM for an inhomogeneous complex-valued configuration with an imperfect interface condition.

2 Problem statement and formulation

2.1 Problem description and strong form

We consider a two-media configuration for the sake of simplicity to formulation our problem as illustrated in. The domain Ω (bounded by $\partial\Omega_1$ and $\partial\Omega_2$)is partitioned into subdomain Ω_1 and Ω_2 to represent classical acoustic fluid such as air and porous layers with $\Omega_1 \cup \Omega_2 = \Omega$ and $\Omega_1 \cap \Omega_2 = \emptyset$. The thin film is condensed as a non-thickness interface Γ^* that $\Gamma^* = \partial\Omega_1 \cap \partial\Omega_2$ with the unit outward normal $\mathbf{n}$ from Ω_1 to Ω_2.

The bulk parts for classical acoustic and porous media are both considered as fluid, the typical Helmholtz equation is thus used to represent the sound propagation. With regard to the resistive thin film, the pressure jump model [7] takes its effects into consideration. There exists a pressure difference when sound wave travel across a film that reads as :

$$[\![p]\!] = \sigma d\bar{v} \tag{1}$$

where $[\![\bullet]\!]$ is the jump operator $[\![\bullet]\!] = \bullet^1 - \bullet^2$ at the interface Γ^*. σ, d and $\bar{v}$ represent the flow resistivity and thickness of films, mean flow velocity at the interface, respectively. The flow velocity is considered conservative as the thickness of films is small compared to the entire domain as :

$$\bar{v} = -\frac{1}{\mathrm{j}\omega\rho_1}\frac{\partial p}{\partial n} = -\frac{1}{\mathrm{j}\omega\rho_2}\frac{\partial p}{\partial n} \tag{2}$$

where ρ_i are the density of fluid in each subdomain, of which the porous materials is in complex number.

Combing the Helmholtz equation for bulk parts and the interface conditions eqs. (1) and (2) and Neumann condition on the boundaries, the strong form of considered boundary value problem reads as :

$$\sum_{i=1}^{2}\left(\nabla^2 p + k_i^2 p\right) = 0, \quad \text{in } \Omega_1 \cup \Omega_2 \tag{3a}$$

$$\frac{\partial p}{\partial n} = g_N, \quad \text{on } \partial\Omega_1 \cup \partial\Omega_2 \tag{3b}$$

$$[\![p]\!] - \sigma d\bar{v} = 0, \quad \text{on } \Gamma^* \tag{3c}$$

$$[\![\frac{1}{\rho}\frac{\partial p}{\partial n}]\!] = 0, \quad \text{on } \Gamma^* \tag{3d}$$

in which k_i are wave numbers, which could be also complex number when representing porous materials. It is worth noticing that the discontinuity of pressure at the interface depends on its gradient, which corresponds to a generalized robin interface condition.

2.2 Weak formulation

We derive here the variational formulation that is able to impose the robin-type interface condition of a discontinuity weakly. Multiplying a test function q for pressure and $1/\rho_i$ to account for interface condition, a weak form of problem eq. (3) is cast as following : find $p \in \mathcal{U}$, $\mathcal{U} := u \in H^1(\Omega)$ such that

$$a(q,p) = \ell(q), \quad \forall q \in \mathcal{V} \tag{4}$$

and the bilinear form a is given as :

$$a(q,p) := a_0(q,p) + a_I(q,p) \tag{5}$$

with

$$a_0(q,p) := \sum_{i=1}^{2}\left(\int_{\Omega_i}\frac{1}{\rho}\boldsymbol{\nabla} q\boldsymbol{\nabla} p\mathrm{d}\Omega - \int_{\Omega_i}\frac{\omega^2}{K_i}q\,p\mathrm{d}\Omega\right) \tag{6a}$$

$$a_I(q,p) := -\int_{\Gamma^*}\frac{q_1}{\rho_1}\frac{\partial p}{\partial n}d\Gamma + \int_{\Gamma^*}\frac{q_2}{\rho_2}\frac{\partial p}{\partial n}d\Gamma \tag{6b}$$

where the $K = \rho c^2$ denotes compressibility of the fluid. With the interface conditions :

$$\frac{1}{\rho_i}\frac{\partial p}{\partial n} = -\mathrm{j}\omega\bar{v} = -\frac{\mathrm{j}\omega}{\sigma d}[\![p]\!] \tag{7}$$

the bilinear form is rewritten as following :

$$a(q,p) := a_0(q,p) + \frac{\mathrm{j}\omega}{\sigma d}\int_{\Gamma^*}[\![q]\!][\![p]\!]d\Gamma \tag{8}$$

Herein, the variational formulation for the problem stated in eq. (3) is obtained. However, this formulation can not treat a vanished interface that $\sigma d = 0$, a small value σd could degrade the conditioning of discrete system as penalty method for enforcing boundary conditions. In addition, with the discretization of X-FEM, the big ratio of "cut-element" leads to an ill-conditioning system that needs to be controlled and stabilized. Therefore, development of a stable variational formulation to tackle these limitations is main focus for the following work.

3 X-FEM discretization and Nitsche's discrete formulation

As the robin-type interface condition yields a strong discontinuity of pressure, the field needs to be enriched to represent such solution as : $p^h \in \mathcal{U}^h \subset \mathcal{U}$

$$p^h(\mathbf{x})\Big|\Omega_e = \sum_i^n N_i(\mathbf{x}) p_i + \sum_j^{n_{enr}} N_j(\mathbf{x}) H(\mathbf{x}) a_j \tag{9}$$

where $N_i(\mathbf{x})$ and $N_j\mathbf{x}$ are the standard shape functions, $H(\mathbf{x})$ represents the generalized Heaviside function which introduces the strong discontinuity in the approximation. a_j are the additional degree of freedoms at the interface.

The interface within our discretization is implicitly defined by level-set function. The mesh for approximation and interpolating geometry of the interface are separated with the aide of technique proposed in the work [12] to improve the convergence for high order approximation.

To overcome the aforementioned numerical issues when solving the weak form eq. (4), the interface operator defined in eq. (8) modified within the X-FEM. We first re-defiene a mean numerical velocity :

$$\bar{v} := -\frac{1}{j\omega}\left\langle \frac{1}{\rho}\frac{\partial p}{\partial n}\right\rangle_\nu \tag{10}$$

where $\langle \bullet \rangle_\nu = \nu(\bullet) + (1-\nu)(\bullet)$ represents the average operator. The value of weighting parameter ν depends on the materials properties on two sides of the interface and the cut ratio of X-FEM element. The value of ν is turned out to be critical for the conditioning of the discrete matrix and is determined in a robust way in [13, 14].

The bilinear form with such mean notation in discrete space $\mathcal{U}^h$is written as :

$$a(q^h, p^h) := a_0(q^h, p^h) - \int_{\Gamma^*} [\![p^h]\!] \left\langle \frac{1}{\rho}\frac{\partial p}{\partial n}\right\rangle_\nu d\Gamma \tag{11}$$

then, addition and subtraction of a term $\frac{\sigma d}{j\omega}\left\langle \frac{1}{\rho}\frac{\partial q}{\partial n}\right\rangle_\nu \left\langle \frac{1}{\rho}\frac{\partial p}{\partial n}\right\rangle_\nu$, we get :

$$a_0(q^h, p^h) - \int_{\Gamma^*} \left\langle \frac{1}{\rho}\frac{\partial p^h}{\partial n}\right\rangle_\nu \left([\![q^h]\!] + \frac{\sigma d}{j\omega}\left\langle \frac{1}{\rho}\frac{\partial q_h}{\partial n}\right\rangle_\nu \right) d\Gamma + \int_{\Gamma^*} \frac{\sigma d}{j\omega}\left\langle \frac{1}{\rho}\frac{\partial q^h}{\partial n}\right\rangle_\nu \left\langle \frac{1}{\rho}\frac{\partial p^h}{\partial n}\right\rangle_\nu d\Gamma \tag{12}$$

at the end, the above formulation is symmetrized and stabilized :

$$\begin{aligned} a_0(q_h, p_h) &- \int_{\Gamma^*} \left\langle \frac{1}{\rho}\frac{\partial p_h}{\partial n}\right\rangle_\nu \left([\![q_h]\!] + \frac{\sigma d}{j\omega}\left\langle \frac{1}{\rho}\frac{\partial q_h}{\partial n}\right\rangle_\nu \right) d\Gamma - \int_{\Gamma^*} \left\langle \frac{1}{\rho}\frac{\partial q_h}{\partial n}\right\rangle_\nu \left([\![p_h]\!] + \frac{\sigma d}{j\omega}\left\langle \frac{1}{\rho}\frac{\partial p_h}{\partial n}\right\rangle_\nu \right) d\Gamma \\ &+ \int_{\Gamma^*} \frac{\sigma d}{j\omega}\left\langle \frac{1}{\rho}\frac{\partial q_h}{\partial n}\right\rangle_\nu \left\langle \frac{1}{\rho}\frac{\partial p_h}{\partial n}\right\rangle_\nu d\Gamma + \lambda \int_{\Gamma^*} \left([\![q_h]\!] + \frac{\sigma d}{j\omega}\left\langle \frac{1}{\rho}\frac{\partial q_h}{\partial n}\right\rangle_\nu \right) \left([\![p_h]\!] + \frac{\sigma d}{j\omega}\left\langle \frac{1}{\rho}\frac{\partial p_h}{\partial n}\right\rangle_\nu \right) \end{aligned} \tag{13}$$

where λ can be defined as a function of element size and interface law similar as in [9, 11] :

$$\lambda = \left(\frac{h}{\gamma} + \frac{\sigma d}{j\omega} \right)^{-1} \tag{14}$$

this value of λ determines the continuity and stability (coercivity) of resulting linear discrete system (can be proven).

4 Results and Discussion

We implement the variational formulation eq. (13) within X-FEM and applied it for a cylinder scattering problem as illustrated in fig. 1 The two subdomain are air and a porous material modelled as equivalent fluid with JCA parameters. Exact fluid velocity are imposed on four outer boundaries as Neumann conditions. The interface for modelling thin film is defined by level-set function using a refined sub-grid mesh. As seen that the elements cut by the interface are in irregular way, which may take place

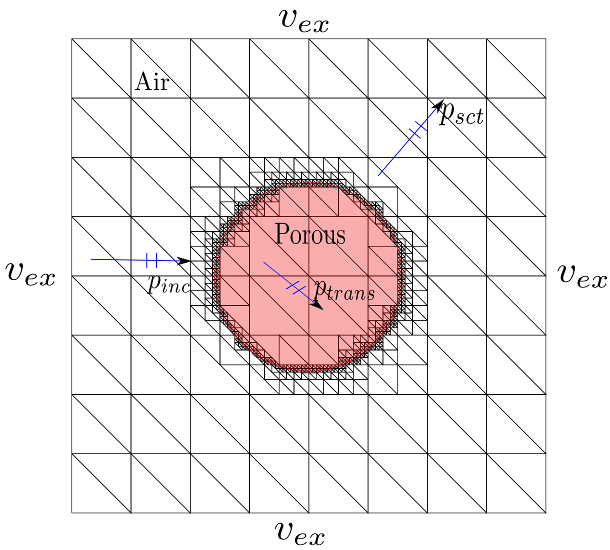
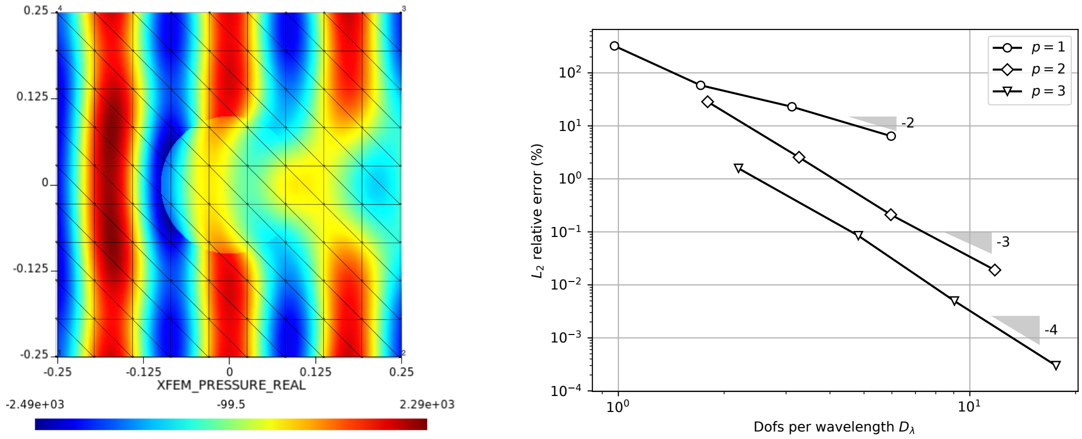


FIGURE 1 – Two media configuration



(a) (b)

FIGURE 2 – (a) Example of solution for fluid pressure ; (b) The associated convergence

some numerical issues for certain formulations. Such mesh is therefore allow us to test the robustness and stability of the proposed method.

Figure 2(a) gives a result of solution where a discontinuity of pressure field is observed at the location of the interface, which is the effect of pressure jump model. The h-convergence of such problem is conducted using five gradual increasing meshes under four polynomial degree compared with analytical solution. Figure 2(b) shows that most curves are close to the optimal rate $O(h^{-(p+1)})$ and are able to achieve a high accuracy.

These results verify the implementation of the proposed formulation behave greatly in X-FEM space and the modified Gauss integration works well on partitioned elements in high order interpolation. The proposed approach is demonstrated efficient and stable for acoustic analysis of the porous structures, in which a strong discontinuity of the pressure is considered. For the sake of brevity, only the application to the fluid-fluid coupling is presented in this summary. We have also successfully extend Nitsche's formulation for fluid-Biot coupling where a fluid-solid interaction are involved.

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